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WALRASIAN BARGAINING

Muhamet Yildiz, MIT

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Abstract

Given any two-person economy, we consider a simple alternating-offer bargaining game with complete information where the agents offer prices. We provide conditions under which the outcomes of all stationary subgame-perfect equilibria converge to the Walrasian equilibrium (the price and the allocation) as the agents become infinitely patient. Therefore, price-taking behavior can be achieved with only two agents.

Key Words: Bargaining, Competitive equilibrium, Coase Conjecture
JEL Numbers: C73, C78, D41.

1 Introduction

It is commonly believed that, in a sufficiently large, frictionless economy, trade results in an approximately competitive (Walrasian) allocation. In fact, the core of such an economy consists of the approximately Walrasian allocations. Moreover, in a bargaining model with a continuum of patient agents, Gale (1986) shows that we will have a competitive allocation in any stationary subgame-perfect equilibrium (henceforth, SSPE). In contrast, when there are only two agents, it is believed that we will have a bilateral monopoly case, and the outcome will be indeterminate (Edgeworth, 1881). Accordingly, in that case, the core is very large, and the SSPE outcome will typically differ from the competitive allocation in the usual bargaining models, in which the agents are allowed

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to offer any allocation.[1] In this paper, we analyze a simple two-agent bargaining model where the agents offer prices. We show that, under certain conditions, as the agents' discount rates approach to 1, the SSPE allocations and the prices converge to the competitive allocation and the price, respectively. Therefore, the Walrasian equilibrium may have stronger foundations than commonly thought.

Considering a pure-exchange economy with only two agents, we analyze the following alternating-offer bargaining game with complete information: Agent 1 offers a price. Agent 2 either demands a feasible trade at that price, or rejects the offer. If he demands a trade, the demanded trade is realized, and the game ends. If he rejects the offer, we proceed to the next date, when Agent 2 offers a price, and Agent 1 either demands a feasible trade or rejects the offer. This goes on until they reach an agreement. Each agent's utility function is normalized by setting to 0 at the initial endowment. Agents cannot consume their goods until they reach an agreement, and each discounts the future with a constant discount rate.

Notice that we restrict the proposer to offer prices, and allow the other agent to choose the amount of trade at that price. For instance, in the case of wage bargaining, if the union sets the wage, the firm has right to choose how much labor to hire. Likewise, if the firm sets the wage, the workers or the union have the right to choose the amount of labor they provide. In contrast, in the standard models, the proposer is allowed to offer any allocation, while the other agent can only accept or reject the offer. This is the only major difference.

When the agents are restricted to making price offers, while their trading partners are allowed to choose the amount of trade at the offered prices, price-taking behavior emerges. To see this, consider an economy with two goods. When an agent i accepts a price, the game ends, hence he demands the optimal trade for i at that price. Effectively, this restricts the other agent to offer a payoff vector on *the offer curve of agent i*, the payoff vectors that can be achieved when i is a price-taker. Moreover, when the discount rates are close to 1, in any SSPE in which all the offers are accepted, both agents are approximately indifferent between the allocations today and tomorrow. Note that the payoffs associated with these allocations are on the offer curves of two distinct agents. Therefore, the payoffs in such a SSPE must be very close to an intersection of the offer

[1] For instance, as the common discount rate approaches to 1, the SSPE outcome in Rubinstein (1982) converges to the Nash (1950) bargaining solution (Binmore, Rubinstein, and Wolinsky, 1987), which is different from the Walrasian outcome, as noted by Binmore (1987). (Rubinstein's model is more general and abstract, but the set of feasible payoffs is typically taken to be the set of materially feasible payoffs, allowing the agents to offer any allocation.)

curves. Of course, each Walrasian payoff-vector is at such an intersection. Moreover, in many canonical economies with a unique Walrasian equilibrium, the Walrasian payoff-vector is the only intersection (of the offer curves) in the relevant region. Therefore, for such economies, the payoffs in these SSPE must be converging to the Walrasian payoffs. Moreover, the Walrasian payoffs are obtained only at the Walrasian allocation — as we have strictly quasi-concave utility functions. By continuity, this shows that the allocation at these SSPE also converge to the Walrasian allocation. Finally, when the utility functions are continuously differentiable, there exists a continuous inverse-demand function, therefore the prices in these SSPE also converge to the Walrasian price.

We further show that in any other possible SSPE, an agent must act as a natural monopoly: he must offer his monopoly price, and it must be accepted. In order for this to be an equilibrium for large values of discount rates, the monopoly outcome must be Pareto-optimal under the constraint that one of the agents must be a price-taker. Ruling out the case that a monopoly is efficient in this sense, we conclude that all SSPE allocations (and the prices) converge to the Walrasian allocation (and the price) of the static pure-exchange economy at hand.

This limiting behavior is strikingly different from that of the Rubinstein (1982) model where agents can offer any allocation. In that model, for a quasi-linear economy, how the agents share the gains from trade in the limit is solely determined by the frequencies the agents make offers, or the logarithmic ratio of the discount rates (as they approach to 1). All these are irrelevant for our limit, the Walrasian allocation of the static economy. Therefore, our small change in the allocation of the rights (embedded in the bargaining procedure) has a profound impact on the bargaining outcome, while the bargaining outcome under the new allocation of rights is very robust to other procedural details.

Our result is related to the literature on the Coase Conjecture. In the context of a durable-good monopoly, Coase (1972) argued that, if a monopolist cannot commit to a price, then he will price his good at the marginal cost. (For the consumers expects the monopolist to lower the prices in the future. In a sense, the monopolist would be competing with its future-selves.) In an asymmetric information model, Gul, Sonnenschein, and Wilson (1986) and Fudenberg, Levine, and Tirole (1985) have shown that this conjecture is true in any SSPE, as the discount rate approaches to 1. In this formalization, it is essential that the monopolist does not know the buyers' valuation. If the buyers' valuations were common knowledge, the monopolist would charge each buyer his valuation. Our result extends Coase's notion to a two-person economy without asymmetric information. Now, there are two agents (and their future selves) setting prices. The

outcome becomes Walrasian as the discount rates approach to 1.

There is a large literature on the relation between the bargaining outcome an the competitive equilibria (see Osborne and Rubinstein (1990) for a survey). To the best of my knowledge, this literature assumes a very large economy, except for Rubinstein and Wolinsky (1990). Rubinstein and Wolinsky (1990) obtain a similar result to that of Gale (1986 and 1987) with only finitely many agents. However, they consider only a special economy in which the core consists of the competitive allocations.

In the next section, we lay out our model. In Section 3, we explain why the SSPE outcome in the standard model is unrelated to the competitive outcome, and why our theorem is still true. In Section 4, we derive our result; the proofs of lemmas are relegated to the Appendix. Section 5 contains some counter examples that show that our assumptions are not superfluous. Section 6 concludes.

2 Model

Let $X = \mathbb{R}^n_+$ be a commodity space with n goods, and P be the set of all price vectors $p = (p^1, \ldots, p^n) \in \mathbb{R}^n_{++}$ where Good 1 is taken to be the numéraire, i.e., $p^1 = 1$, and $p^k > 0$ for all k.[2] (Henceforth, we will simply say prices instead of price vectors.) Consider a 2-person economy $e = ((u_1, \bar{x}_1), (u_2, \bar{x}_2))$, where $\bar{x}_i \in X$ and $u_i : X \to R$ are the initial endowment and the utility function of agent i, respectively, for each $i \in N$. Assume that u_i is strictly quasi-concave, continuous, monotonically-increasing, and $u_i(\bar{x}_i) = 0$ for each $i \in N = \{1, 2\}$. Write $w = \bar{x}_1 + \bar{x}_2$.

We wish to understand the relation between the Walrasian equilibrium of e and the stationary subgame-perfect equilibria of the following perfect-information game $G(\delta_1, \delta_2)$, where these two agents bargain alternatingly offering prices. Let $T = \{0, 1, 2, \ldots\}$ be the set of all dates. At date $t = 0$, Agent 1 offers a price $p_1 \in P$. Agent 2 either demands a consumption $x_2 \in \{x \in X | (x - \bar{x}_2) \cdot p_1 = 0, x \leq w\}$ or rejects the offer. (Note that x_2 is feasible and can be reached by reallocating the endowments at price p_1.) If she demands x_2, the game ends yielding the payoff vector $\left(\delta_1^t u_1(w - x_2), \delta_2^t u_2(x_2)\right)$, which is associated with the allocation $(w - x_2, x_2)$. If she rejects, we proceed to the next date. At $t = 1$, Agent 2 offers a price p_2, and Agent 1 either demands $x_1 \in \{x \in X | (x - \bar{x}_1) \cdot p_2 = 0, x \leq w\}$, when the game ends yielding the payoff vector $\left(\delta_1^t u_1(x_1), \delta_2^t u_2(w - x_1)\right)$, or rejects the offer, when we proceed to the next date,

[2] We write $\mathbb{R}$ for the set of real numbers, $\mathbb{R}^n_+$ for the non-negative orthant of a n-dimensional Euclidean space, $\mathbb{R}^n_{++}$ for the interior of $\mathbb{R}^n_+$.

when Agent 1 offers a price again. This goes on indefinitely until they reach an agreement. If they never reach an agreement, each gets 0. We are interested in the case when $(\delta_1, \delta_2) \in (0,1)^2$ approaches to $(1,1)$.

Any *stationary subgame-perfect equilibrium* (henceforth, SSPE) is represented by a list $(\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2)$ where, for each $i \in N$, $\hat{p}_i \in P$ is the price that the agent i offers whenever he is to make an offer, and $\hat{x}_i : P \to X \cup \{\text{Reject}\}$ is the function that determines his responses: when offered a price p_j, the agent i demands $\hat{x}_i(p_j)$ if $\hat{x}_i(p_j) \in X$, and rejects the offer if $\hat{x}_i(p_j) = \text{Reject}$.

Basic Definitions and Assumptions For each $i \in N$, define the (constrained) *demand function* $D_i : P \to X$ by setting

$$D_i(p) = \arg\max \{u_i(x) \mid (x - \bar{x}_i) \cdot p \leq 0, \ x \leq w, \ x \in X\}$$

at each $p \in P$. Since u_i is continuous and strictly quasi-concave, D_i is a well-defined function. Since u_i is continuous, by the Maximum Theorem, D_i is also continuous. A (constrained) *Walrasian equilibrium* is any pair $(p, (x_1, x_2))$ of a price $p \in P$ and an allocation $(x_1, x_2) \in X^2$ such that $x_i = D_i(p)$ for each i, and $x_1 + x_2 = w$.

Assumption 1 *There exists a unique Walrasian equilibrium* $(p^*, (x_1^*, x_2^*))$; $(x_1^*, x_2^*) \neq (\bar{x}_1, \bar{x}_2)$, *and* x_i^* *is in the interior of* X *for each* $i \in N$.

Together with strict quasi-concavity, the assumption that $(x_1^*, x_2^*) \neq (\bar{x}_1, \bar{x}_2)$ guarantees that $u_i(x_i^*) > 0$ for each i. In that case, the initial allocation is not Pareto optimal, hence there are gains from trade. The assumption that x_i^* is in the interior of X for each i is made only to make sure that $(p^*, (x_1^*, x_2^*))$ is an "unconstrained" Walrasian equilibrium.

Define the *offer curves* of Agents 1 and 2 as

$$OC_1 = \{(u_1(D_1(p)), u_2(w - D_1(p))) \mid p \in P\}$$

and

$$OC_2 = \{(u_1(w - D_2(p)), u_2(D_2(p))) \mid p \in P\},$$

respectively. OC_i is the set of all utility pairs that can be reached by offering a price to agent i, who will then maximize his payoff given the price. In general, given any payoff v_i for an agent i, there might be multiple pairs (v_1, v_2) in OC_i. In that case, if the other agent j is to choose between these pairs by offering different prices, he will choose the

pair with maximum v_j. To formulate this, define functions U_1 and U_2 (on appropriate domains) by

$$U_1(v_2) = \max\{v_1 | (v_1, v_2) \in OC_2\}$$

and

$$U_2(v_1) = \max\{v_2 | (v_1, v_2) \in OC_1\},$$

respectively. Note that the Walrasian payoff-vector $v^* = (u_1(x_1^*), u_2(x_2^*))$ is in $OC_1 \cap OC_2$, and satisfies the equations

$$u_1(x_1^*) = U_1(u_2(x_2^*)) \text{ and } u_2(x_2^*) = U_2(u_1(x_1^*)). \tag{1}$$

That is, the graphs of U_1 and U_2 (which are typically the offer curves themselves) intersect each other at the Walrasian payoff-vector. For our main result, we will assume that this is the only intersection in the relevant region (see Assumption 4 below).

Throughout the paper, we will make the following assumption, which is satisfied by many economies, such as the Cobb-Douglas economies in the Edgeworth box.

Assumption 2 *For each $i \in N$, U_i is continuous and single-peaked.*

Since U_i is single-peaked and cannot be monotonically increasing, it is maximized at some $\bar{v}_j \in \mathbb{R}$. (Note that U_i is strictly increasing at any $v_j < \bar{v}_j$ and strictly decreasing at any $v_j > \bar{v}_j$.) By a *monopoly price of an agent* i, we will mean any price $\bar{p}_i \in P$ with $u_j(D_j(\bar{p}_i)) = \bar{v}_j$ and $u_i(w - D_j(\bar{p}_i)) = U_i(\bar{v}_j)$.

3 An Example

In this section, using a canonical example, we will explain our formulation, and show how the Walrasian outcome typically differs from the subgame-perfect equilibrium (henceforth SPE) outcome in Rubinstein's model where the agents bargain by offering allocations. We will then explain why our theorem is true.

Consider a quasi-linear economy $e = ((u_1, (0, 1)), (u_2, (M, 0)))$ with two goods where $u_1(m, y) = m + y^\alpha - 1$, $u_2(m, y) = m + y^\alpha - M$, $\alpha \in (0, 1)$, and $M > 0$. Since we have quasi-linear utility functions, we will take $X = \mathbb{R} \times \mathbb{R}_+$, allowing negative amounts of the first good — money.

For $\alpha = 0.5$, the offer curves are plotted in Figure 1. Notice that both U_1 and U_2 are single peaked and continuous. The offer curves OC_1 and OC_2 are simply the graphs of

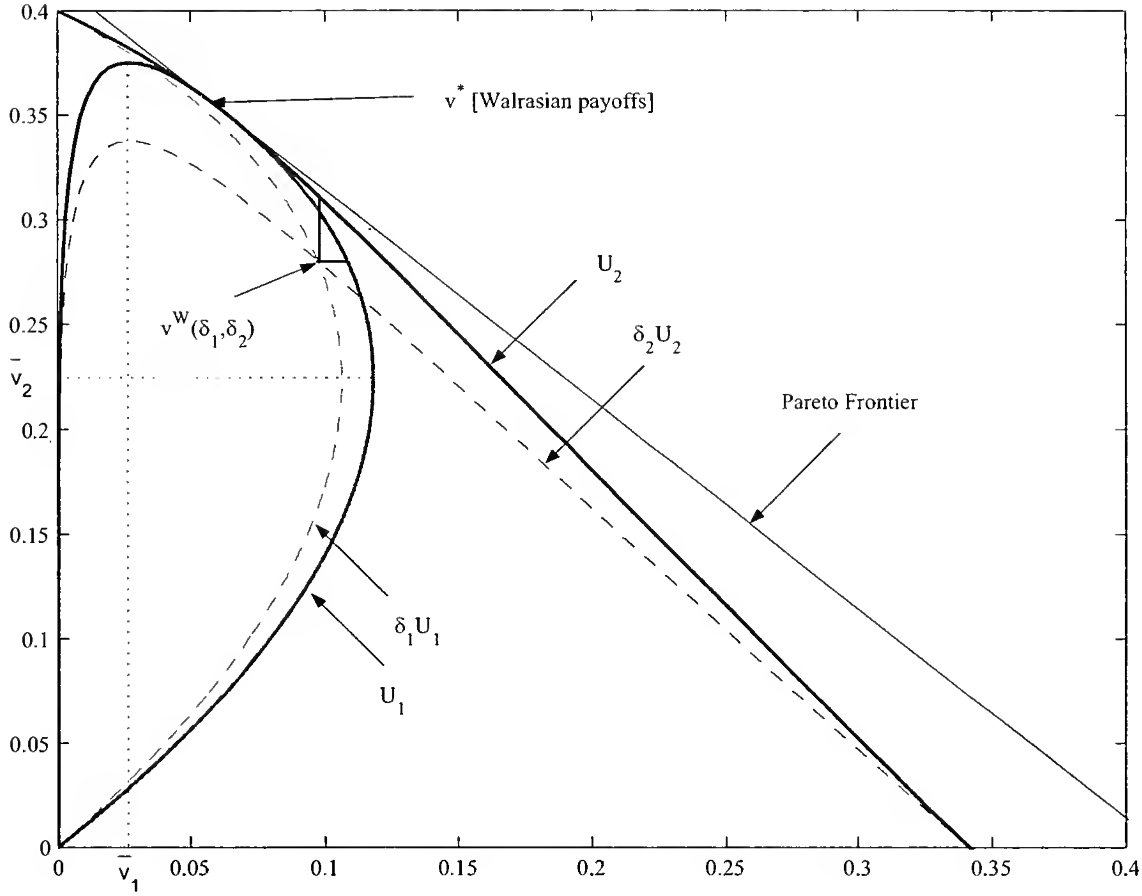

Figure 1: Offer curves for $\alpha = 1/2$, $\delta_1 = \delta_2 = 0.9$.

U_2 and U_1, respectively. The Walrasian payoffs are at the unique point where the offer curves intersect each other. Notice also that the Walrasian payoffs are very asymmetric.

One can compute that the Walrasian price is $p^* = (1, \alpha 2^{1-\alpha})$, yielding the Walrasian payoff vector

$$v^* = \left(\frac{1+\alpha}{2^\alpha} - 1, \frac{1-\alpha}{2^\alpha}\right).$$

Hence, at the Walrasian equilibrium, the ratio of Agent 2's share to Agent 1's share is $(1-\alpha) / (1+\alpha-2^\alpha)$, determined by α.

On the other hand, we have a transferable utility case: in any Pareto-optimal allocation, the payoffs add up to $2^{1-\alpha} - 1$. Then, the unique SPE outcome in Rubinstein's model is

$$v^R(\delta_1, \delta_2) = \left[2^{1-\alpha} - 1\right] \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right).$$

As $(\delta_1, \delta_2) \to (1,1)$ at the rate $r = \log(\delta_1) / \log(\delta_2)$,

$$v^R(\delta_1, \delta_2) \to \left[2^{1-\alpha} - 1\right] \left(\frac{1}{1+r}, \frac{r}{1+r}\right).$$

Therefore, in the limit, the ratio of Agent 2's share to Agent 1's share is r, determined solely by the discount rates, or the frequencies at which the agents make offers. This well-known fact implies that, in the limit, the SPE payoffs in Rubinstein's model cannot possibly be related to the Walrasian payoffs, which are determined by α. In particular, when there is a common discount rate, in the limit, the SPE distributes the gains from trade equally, while the Walrasian payoffs are very asymmetric.

Now consider the bargaining procedure in $G(\delta_1, \delta_2)$. Lemma 1 below establishes that, whenever an agent i accepts a price p_j, he demands the optimal consumption $D_i(p_j)$ — for the game ends there, and thus our bargaining procedure can be considered as the Rubinstein's bargaining model where each agent is restricted to offer payoffs on the other agent's offer curve. Therefore, as in Rubinstein (1982), there is a SPE that is determined by the intersection $v^W(\delta_1, \delta_2)$ of the graphs of $\delta_1 U_1$ and $\delta_2 U_2$ (see Figure 1). In this SPE, an agent i accepts an offer iff he gets at least $v_i^W(\delta_1, \delta_2)$, and the other agent j offers a price that gives $v_i^W(\delta_1, \delta_2)$ to i and $U_j\left(v_i^W(\delta_1, \delta_2)\right)$ to j. But, since v^* is the unique intersection of the graphs of U_1 and U_2, as $(\delta_1, \delta_2) \to (1,1)$,

$$v^W(\delta_1, \delta_2) \to v^*.$$

In the limit, the SPE payoff-vector itself is v^*. This convergence is independent of the rate at which the discount rates go to 1. This also suggests that this convergence to

the Walrasian equilibrium may hold even if the agents would make offers at different frequencies.

As one can easily see from Figure 1, the changes in the agents' discount rates do change the bargaining outcome — in a predictable way. When we increase the discount rate of an agent keeping everything else constant, his equilibrium payoff (if anything) increases, while the other agent's equilibrium payoff (if anything) decreases, provided that there is a unique SSPE.

Notice in Figure 1 that $(\bar{v}_1, U_2(\bar{v}_1))$ and $(U_1(\bar{v}_2), \bar{v}_2)$ are below the graphs of U_1 and U_2, respectively. That is, if an agent offers his monopoly price, then (for large values of discount rates) the other agent can reject that offer and make a Pareto-improving counter-offer. Under this condition, for large values of discount rates, we show that there cannot be other SSPE. Therefore, when this condition and the unique intersection property of U_1 and U_2 hold, all the SSPE converge to the Walrasian outcome — as in this example.

4 Theorem

In this section, we will describe the SSPE of game $G(\delta_1, \delta_2)$. We will then provide sufficient conditions under which the allocations and the prices at all SSPE converge to the Walrasian allocation and the price, respectively, as $(\delta_1, \delta_2) \to (1, 1)$.

Our first lemma describes the basic properties of SSPE.

Lemma 1 *Given any $\delta_1, \delta_2 \in (0,1)$, any $i \neq j \in N$, any SSPE $(\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2)$ of $G(\delta_1, \delta_2)$, under Assumption 2, the following are true:*

1. $\hat{x}_j(p_i) \in \{D_j(p_i), \text{Reject}\}$ *for all* $p_i \in P$;
2. *if* $\hat{x}_j(\hat{p}_i) \neq \text{Reject}$, *then* $u_i(w - D_j(\hat{p}_i)) = U_i(u_j(D_j(\hat{p}_i)))$;
3. *if* $\hat{x}_j(\hat{p}_i) \neq \text{Reject}$, *then* $u_j(D_j(\hat{p}_i)) \geq \bar{v}_j$.

Part 1 states that, if an agent accepts a price, he demands his optimal consumption at that price, for the game ends there. Hence, in terms of equilibria, our game is equivalent to a bargaining game where each agent is restricted to offer a payoff vector in the other agent's offer curve. In that case, each agent i offers a point on the graph of U_i —hence the second part. The proof of this part uses the continuity of U_i and the availability of the prices that allow the other agent to demand consumptions better

than his equilibrium demand. Part 3 simply states that each agent's offer is at least as generous as his monopoly price.

Our next lemma lists some necessary conditions for a SSPE. This is the main step towards proving our Theorem. The basic argument is the following. Under Assumption 2, if an agent j is offering a price $\hat{p}_j$ that will be accepted and that allows the other agent i to obtain a higher payoff than his continuation value, then $\hat{p}_j$ must be a monopoly price of j. For, otherwise, j would offer a less generous price that would be accepted and would yield a higher payoff for j. This implies that either (i) or (ii) below must hold.

Lemma 2 *Under Assumptions 1 and 2, for any SSPE $(\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2)$ of any $G(\delta_1, \delta_2)$, either (i) or (ii) is true:*

(i) for all distinct $i, j \in N$,

$$\begin{aligned} \hat{x}_i(\hat{p}_j) &= D_i(\hat{p}_j), &(2)\\ u_i(D_i(\hat{p}_j)) &= \delta_i u_i(w - D_j(\hat{p}_i)) = \delta_i U_i(u_j(D_j(\hat{p}_i))); &(3) \end{aligned}$$

(ii) there exist some distinct i and j such that

$$\begin{aligned} \hat{x}_i(\hat{p}_j) &= D_i(\hat{p}_j), &(4)\\ u_i(D_i(\hat{p}_j)) &= \bar{v}_i. &(5) \end{aligned}$$

Condition (i) describes a type of equilibrium: Each equilibrium-offer is accepted, and the equilibrium offers leave the other agents indifferent between accepting (and demanding the optimal consumption) and rejecting the offer. This indifference yields the equation system (3), similar to the equation system that characterizes the equilibrium in Rubinstein (1982). At $\delta_1 = \delta_2 = 1$, this equation system is identical to (1), the system of equations satisfied by the Walrasian prices.

Condition (ii) describes another type of equilibrium: Now, there exists an agent i who always offers his monopoly price, which is accepted by the other agent. The other agent's offer is typically rejected. This is an equilibrium iff there is no point on the offer curve of j that gives each agent at least the continuation value — the discounted value of payoffs when j is a monopoly. In that case, the other agent (i) cannot offer any price that must be accepted by the sequentially-rational agent j.[3]

[3] In a bargaining model where the agents bargain over trades (rather than prices), an agent can always offer the trade that will take place in the next date, guaranteeing the continuation value to each agent, but this is not true in our model.

Lemma 3 *Under Assumptions 1 and 2, for any SSPE* $(\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2)$ *of any* $G(\delta_1, \delta_2)$, *if (ii) of Lemma 2 is true, then there does not exist any* $p_i \in P$ *such that*

$$(\delta_i u_i (w - D_j (p_i)), u_j (D_j (p_i))) > (\bar{v}_i, \delta_j U_j (\bar{v}_i)), \tag{6}$$

i.e., the inequality $\delta_i U_i (\delta_j U_j (\bar{v}_i)) > \bar{v}_i$ *does not hold.*

We will now assume that (6) holds for some price in the limit when $\delta_1 = \delta_2 = 1$, ruling out all the equilibria of the second type for large values of δ_1 and δ_2.

Assumption 3 *For all* $i, j \in N$, *there exists some* $p_i \in P$ *such that*

$$(u_i (w - D_j (p_i)), u_j (D_j (p_i))) > (\bar{v}_i, U_j (\bar{v}_i)).$$

That is, an agent i can offer a price that is better than the monopoly price of the other agent j for both of them, provided that j maximizes his payoff given the price offered by i. In other words, any monopoly is Pareto-inefficient even under the constraint that the agents trade through prices. By continuity, this assumption implies that (6) holds for large values of (δ_1, δ_2).

Lemma 4 *Under Assumption 3, there exists some* $\bar{\delta} \in (0, 1)$ *such that, for all* $\delta_1, \delta_2 \in (\bar{\delta}, 1)$, *and for all* $i, j \in N$, *we have* $\delta_i U_i (\delta_j U_j (\bar{v}_i)) > \bar{v}_i$, *i.e., there exists* $p_i \in P$ *satisfying (6).*

The following is an immediate corollary to Lemmas 2, 3, and 4:

Lemma 5 *Under Assumptions 1-3, there exists some* $\bar{\delta} \in (0, 1)$ *such that, for all* $\delta_1, \delta_2 \in (\bar{\delta}, 1)$, *for all SSPE* $(\hat{p}_1, \hat{x}_1, \hat{p}_2, \hat{x}_2)$ *of* $G(\delta_1, \delta_2)$, *and for all distinct* $i, j \in N$, *we have*

$$\hat{x}_i (\hat{p}_j) = D_i (\hat{p}_j), \tag{7}$$

$$u_i (D_i (\hat{p}_j)) = \delta_i u_i (w - D_j (\hat{p}_i)) = \delta_i U_i (u_j (D_j (\hat{p}_i))). \tag{8}$$

The following assumption guaranties that the graphs of U_1 and U_2 has a unique intersection in the relevant region.

Assumption 4 *There exists a unique* $(v_1, v_2) \geq (\bar{v}_1, \bar{v}_2)$ *such that*

$$U_1 (v_2) = v_1 \text{ and } U_2 (v_1) = v_2.$$

At the Walrasian equilibrium, each agent i gets at least $\bar{v}_i$, and the graphs of U_1 and U_2 intersect each other. Therefore, we have the following lemma.

Lemma 6 *Under Assumption 4, given any $p_1, p_2 \in P$, we have*

$$U_i\left(u_j\left(D_j\left(p_i\right)\right)\right) = u_i\left(D_i\left(p_j\right)\right) \geq \bar{v}_i \qquad (\forall i \neq j \in N)$$

iff $p_1 = p_2 = p^*$.

This gives us our main theorem.

Theorem 1 *Under Assumptions 1-4, if $\left(\hat{p}_1^\delta, \hat{x}_1^\delta, \hat{p}_2^\delta, \hat{x}_2^\delta\right)$ is a SSPE of $G(\delta)$ for each $\delta = (\delta_1, \delta_2) \in (0,1)^2$, then*

$$\lim_{\delta \to (1,1)} \hat{x}_j^\delta\left(\hat{p}_i^\delta\right) = x_j^* \qquad (9)$$

for each distinct $i, j \in N$. Moreover, if u_1 and u_2 are continuously differentiable, then

$$\lim_{\delta \to (1,1)} \hat{p}_1^\delta = \lim_{\delta \to (1,1)} \hat{p}_2^\delta = p^*. \qquad (10)$$

That is, as the discount rates approach to 1, under Assumptions 1-4, the SSPE allocations converge to the unique Walrasian allocation. If the utility functions are continuously differentiable, this implies that the SSPE prices also converge to the Walrasian price. Notice that the Theorem does not claim the existence of SSPE. (Under Assumptions 1-4, one can easily show that there does exist a SSPE.)

Proof. Define $A = \left(D_1(P) \times D_2(P)\right) \cap \left\{(x_1, x_2) \in X^2 | \forall i \in N,\, u_i(x_i) \geq \bar{v}_i\right\}$. Define also the function $f : (0,1)^2 \times A \to \mathbb{R}^2$ by

$$f_i(\delta_1, \delta_2, x_1, x_2) = \delta_i U_i\left(u_j(x_j)\right) - u_i(x_i) \qquad (i \neq j \in N)$$

and the correspondence $\xi : [0,1]^2 \to 2^A$ by

$$\xi(\delta_1, \delta_2) = \left\{(x_1, x_2) \in A | f(\delta_1, \delta_2, x_1, x_2) = 0\right\}.$$

By Lemma 6,

$$\xi(1,1) = \left\{(x_1^*, x_2^*)\right\}.$$

Moreover, since f is continuous, ξ has a closed graph. Since A is compact, this implies that ξ is upper semi-continuous. Hence, given any $\epsilon > 0$, there exists $\hat{\delta} \in (0,1)$ such that, for each $\delta_1, \delta_2 > \hat{\delta}$, for each $(x_1, x_2) \in \xi(\delta_1, \delta_2)$, and for each $j \in N$, we have

$$\left\|x_j - x_j^*\right\| < \epsilon. \qquad (11)$$

On the other hand, by Lemma 5, there exists a $\bar{\delta} \in (0,1)$ such that, for each $\delta = (\delta_1, \delta_2) \in (\bar{\delta}, 1)^2$,

$$\left(\hat{x}_1^\delta\left(\hat{p}_2^\delta\right), \hat{x}_2^\delta\left(\hat{p}_1^\delta\right)\right) \in \xi\left(\delta_1, \delta_2\right). \tag{12}$$

Therefore, by (11) and (12), given any $\epsilon > 0$, there exists $\delta^* \geq \max\{\hat{\delta}, \bar{\delta}\}$ such that, for each $\delta = (\delta_1, \delta_2) \in (\delta^*, 1)^2$, we have $\left\|\hat{x}_j^\delta\left(\hat{p}_i^\delta\right) - x_j^*\right\| < \epsilon$, proving (9).

Towards proving the second part, define $\overset{\circ}{X} = \left\{x \in X | 0 < x^k < w^k \ \forall k \leq n\right\}$, the set of consumption bundles corresponding to the interior allocations. For any distinct $i, j \in N$, if u_i is continuously differentiable, then inverse-demand function $D_i^{-1} : \overset{\circ}{X} \cap D_i(P) \to P$ exists and continuous, where $D_i\left(D_i^{-1}(x_i)\right) = x_i$ for each $x_i \in \overset{\circ}{X} \cap D_i(P)$. But, since $\hat{x}_j^\delta\left(\hat{p}_i^\delta\right) \to x_j^*$, by continuity, $\hat{p}_i^\delta = D_i^{-1}\left(\hat{x}_j^\delta\left(\hat{p}_i^\delta\right)\right) \to D_i^{-1}\left(x_j^*\right) = p^*$. ■

An intuition for the Theorem is the following. In equilibrium, Agent 1 must be maximizing his utility at the price set by Agent 2, and Agent 2 must be maximizing his utility at the price set by Agent 1. The markets clear by definition. As the discount rates converge to 1, one would expect that the prices set by different agents will become similar, and each agent will become indifferent between today and tomorrow. That is, approximately, each agent i is indifferent between he himself maximizing his utility at the price set by the other agent j and the other agent j maximizing her payoff at approximately the same price set by i. Therefore, we must be approximately at a Walrasian equilibrium. (The logic of our proof is clearly different from this intuition, for proving convergence to a Walrasian equilibrium turns out to be more straight-forward than proving that the prices converge to the same price.)

What is the essential aspect of the Walrasian equilibrium — is it competition or price taking? In the tradition of Coase (1972), but relying on different forces, our Theorem suggests that, when the goods are durable, the competition between the selves now and later may be sufficient for the price-taking behavior to emerge, provided that the agents are restricted to trade using prices as in this paper.

In sequential bargaining models, the outcome is determined by which agent makes and offer when. Some authors find this a weakness of the model and adhere to the traditional view that the bargaining outcome should be determined by the property rights — not by the details of the procedure (see Aumann (1987), Nash (1950), and Harsanyi, 1977).[4] Our theorem has two implications on this issue. On the one hand, it establishes that a change in the procedure (that the proposers are required to offer

[4]See also Perry and Reny (1993), who develop a bargaining model with endogenous timing of offers, and Smith and Stacchetti (2001), who analyze the equilibria of the continuous time game that are not determined by the temporal monopoly of the proposer.

prices rather than allocations while the responders are allowed to choose the size of the trade) has a profound effect on the outcome. Therefore, *the allocation of procedural rights matters.* On the other hand, the limiting behavior under the new allocation of the procedural rights does not depend on how the discount rates approach to 1. Hence, one can easily show that it would not depend which agent makes an offer when, provided that each agent makes offers sufficiently frequently. Therefore, *such procedural details do not matter for the limiting behavior* under the new allocation of procedural rights.

Early mechanism design literature had an interest in implementing the Walrasian allocation in a Nash environment (see Hurwicz (1979) and Schmeidler, 1980). Our Theorem establishes that our simple bargaining procedure approximately implements the Walrasian allocation in SSPE under certain conditions, provided that the initial endowments are known by the mechanism designer.

Along with the simplifying assumptions 1 and 2, our Theorem assumes that the monopolies are inefficient even under the constraint that one of the agents must be a price-taker (Assumption 3) and the graphs of U_1 and U_2 have a unique intersection in the relevant region (Assumption 4). Under these assumptions it establishes that SSPE outcome converges to the Walrasian outcome. Are these assumptions superfluous? Are there non-stationary subgame-perfect equilibria? These questions are answered in the next section.

5 Counter-examples

We will now present two examples. The first example shows that, if Assumption 3 fails, there may exist SSPE as described in condition (ii) of Lemma 2, which do not converge to the Walrasian outcome. It also shows that there may be multiple SSPE that converge to the Walrasian equilibrium. Hence, there may be non-stationary subgame-perfect equilibria, based on the SSPE that converge to the Walrasian equilibrium. The second example shows that, when Assumption 4 fails, SSPE may converge to a non-Walrasian outcome.

Example 1 Consider the economy $e = ((u_1, (9,1)), (u_2, (1,9)))$ where $u_1(x,y) = x^{\alpha}y^{1-\alpha} - 9^{\alpha}$ and $u_2(x,y) = x^{1-\beta}y^{\beta} - 9^{\beta}$. In Figure 2, we plot the offer curves for $\alpha = 0.6$, $\beta = 0.1$, and $\delta_1 = \delta_2 = 0.9$. Firstly, notice that $U_1(\bar{v}_2) > \max\{v_1 | U_2(v_1) \geq 0\}$. Hence, for large values of (δ_1, δ_2), we have a SSPE $(\bar{p}_1, \hat{x}_1, \bar{p}_2, \hat{x}_2)$ where $\hat{x}_1(p) = D_1(p)$ iff $u_1(D_1(p)) \geq \delta_1 U_1(\bar{v}_2)$ and $\hat{x}_2(p) = D_2(p)$ iff $u_2(D_2(p)) \geq 0$. (Recall that $\bar{p}_i$ is

the monopoly price of i.) In this SSPE, Agent 1 emerges as a monopoly: he offers his monopoly price $\bar{p}_1$, and it is accepted. There is no price that Agent 1 would accept and that gives positive payoff to Agent 2, so he offers the non-serious price $\bar{p}_2$, which will be rejected. Second, observe that, for $\delta_1 = \delta_2 = 0.9$, the graphs of $\delta_1 U_1$ and $\delta_2 U_2$ intersect each other at two distinct points v and v', yielding two more SSPE. Both of these equilibria converge to the Walrasian outcome.

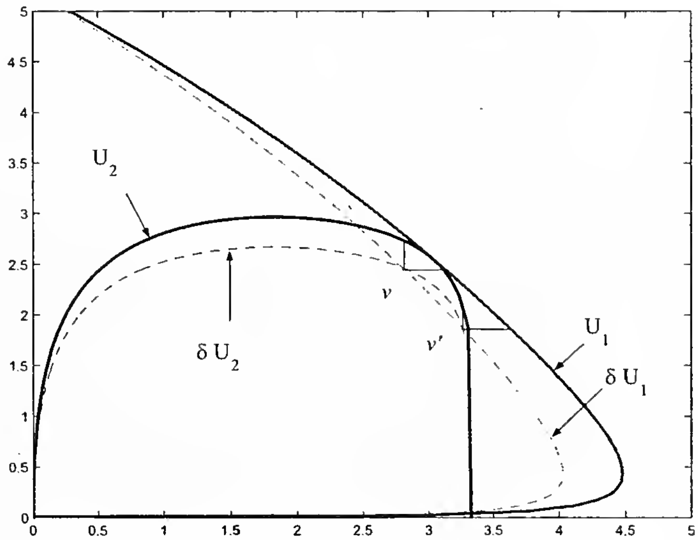

Figure 2: The offer curves for Example 1.

The following example is taken from Sertel and Yildiz (1994). Using this economy, they show that there cannot be an axiomatic bargaining solution that always pick the Walrasian payoff-vectors. Here, we will use the same economy to show that, if the graphs of U_1 and U_2 intersect each other at non-Walrasian payoff vectors, the SSPE can converge to non-Walrasian outcomes.

Example 2 (Sertel and Yildiz, 1994) Consider the economy $e = ((u, (0, 10)), (u, (10, 0)))$ where $u(x, y) = (1/45) \min\{24x + 3y + 15, 9x + 18y, 4x + 23y + 5\} - 1$. The indifference curves and the offer curves (in utility space and in the Edgeworth box) are plotted in Figure 3. The indifference curves are tangent to each other only at the strip around the diagonal, when the common slope of the indifference curves is $-1/2$. Therefore, the unique Walrasian price in this economy is $p^* = (1, 2)$, yielding a unique Walrasian payoff-vector $v^* = (4, 1)$. In the Edgeworth box, the offer curves of Agent 1 and Agent 2 are

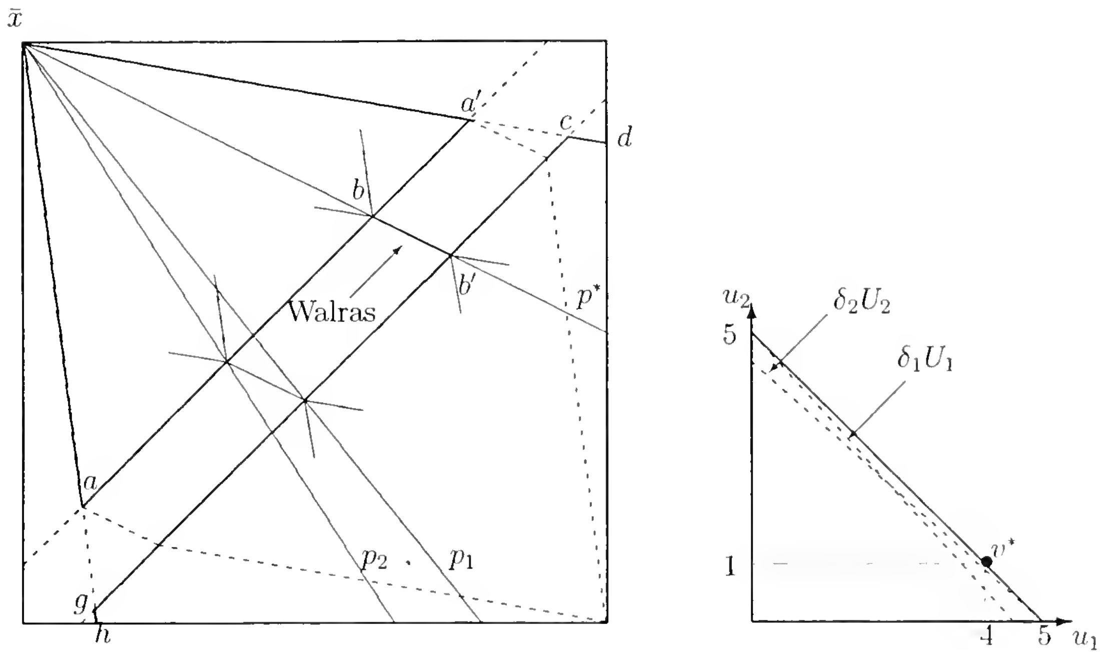

Figure 3: The offer curves for economy e of Example 2 — in the Edgeworth box and in the utility space. (The hyperplanes are indicated by the associated prices.)

the line segments connecting the points $[\bar{x}, a, b, b', c, d]$ and $[\bar{x}, a', b, b', g, h]$, respectively. But in the utility space, the graphs of U_1 and U_2 coincide: we have $U_2(v_1) = 5 - v_1$ and $U_1(v_2) = 5 - v_2$. Then, the SSPE in this economy is as in Rubinstein (1982) for the transferable utility case. For each δ_1, $\delta_2 \in (0, 1)$, there exists a unique SSPE — determined by the intersections of the graphs of $\delta_1 U_1$ and $\delta_2 U_2$ — with payoff vector

$$5\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right).$$

When we fix $\delta_1 = \delta_2 = \delta$ and let $\delta \to 1$, the SSPE payoffs converge to $(5/2, 5/2)$, distinct from the Walrasian payoff-vector $v^* = (4, 1)$. The SSPE prices $\hat{p}_1^\delta$ and $\hat{p}_2^\delta$ converge to $p_1 = (1, 29/37)$ and $p_2 = (1, 7/11)$, respectively.

Even though the prices p_1 and p_2 are distinct, the equilibrium allocation is Pareto-optimal in the limit, for the utility functions are concave but not strictly concave. One can slightly modify the utility functions in this example to construct strictly concave utility functions, yielding the same Walrasian payoffs and the same SSPE. In that case, the allocation in the limit would be Pareto-inefficient, but there would not be a Pareto-improving trade that can be achieved through price taking behavior.

6 Conclusion

Consistent with common sense, the SSPE in usual bargaining models yield the Walrasian allocation in a large economy (Gale, 1986), while they typically yield non-Walrasian outcomes when there are only limited number of agents. In these bargaining models, agents are allowed to offer any trade among the bargaining parties. Here, we present a simple bargaining procedure where the agents are restricted to offer prices, while their trading partners optimize at these prices. As the discount rates go to 1, under certain conditions, the SSPE of this mechanism yield the Walrasian equilibrium. Therefore, the Walrasian equilibrium does not necessarily require a large economy. It simply corresponds to price-taking behavior, which can be achieved even with only two agents.

A Proofs of Lemmas

This appendix contains the proofs of the Lemmas.

Proof. **[Lemma 1]** (Part 1) If $\hat{x}_j(p_i) \neq$ Reject, then we are in a final decision node, hence we must have $\hat{x}_j(p_i) = D_j(p_i)$.

In order to prove Part 2, we will first show that, if $\hat{x}_j(\hat{p}_i) \neq$ Reject, then there exists some $\tilde{p} \in P$ such that

$$u_j(D_j(\hat{p}_i)) < u_j(D_j(\tilde{p}_i)). \tag{13}$$

Assume that $\hat{x}_j(\hat{p}_i) \neq$ Reject, i.e., $\hat{x}_j(\hat{p}_i) = D_j(\hat{p}_i)$. If $u_j(D_j(\hat{p}_i)) = 0$, $\tilde{p}_i = p^*$ satisfies (13). Assume that $u_i(D_j(\hat{p}_i)) > 0$. Hence, $D_j(\hat{p}_i) \neq \bar{x}_j$, and $w - D_j(\hat{p}_i) \neq \bar{x}_i$. Now, since i offers $\hat{p}_i$, it must be that

$$u_i(w - D_j(\hat{p}_i)) \geq 0. \tag{14}$$

By the Separating-Hyperplane Theorem, there exists a price $\tilde{p}_i$ such that $D_i(\tilde{p}_i) = \bar{x}_i$.For any $x \neq \bar{x}_i$ with $\tilde{p}_i \cdot (x - \bar{x}_i) \leq 0$, $u_i(x) < 0$. Together (14), this implies that $0 > \tilde{p}_i \cdot (w - D_j(\hat{p}_i) - \bar{x}_i) = \tilde{p}_i \cdot (\bar{x}_j - D_j(\hat{p}_i))$, i.e., $\tilde{p}_i \cdot (D_j(\hat{p}_i) - \bar{x}_j) < 0$. Since u_i is monotonically increasing, this implies (13).

(Part 2) Assume that $\hat{x}_j(\hat{p}_i) \neq$ Reject, i.e., $\hat{x}_j(\hat{p}_i) = D_j(\hat{p}_i)$. But, by stationarity, the continuation values do not depend which offers are rejected, hence, for any p_i with $u_j(D_j(p_i)) > u_j(D_j(\hat{p}_i))$, we must have $\hat{x}_j(p_i) \neq$ Reject, and thus $\hat{x}_j(p_i) = D_j(p_i)$. Suppose that $u_i(w - D_j(\hat{p}_i)) < U_i(u_j(D_j(\hat{p}_i)))$. Since U_i is continuous, by (13), there then exists some $\epsilon > 0$ and some price p' such that $U_i(u_j(D_j(\hat{p}_i)) + \epsilon) > u_i(w - D_j(\hat{p}_i))$

and $u_j(D_j(p')) = u_j(D_j(\hat{p}_i)) + \epsilon$. Now, if i offers p', it will be accepted, yielding higher payoff $U_i(u_j(D_j(\hat{p}_i)) + \epsilon)$, a contradiction.

(Part 3) Suppose that $\hat{x}_j(\hat{p}_i) \neq$ Reject and $u_j(D_j(\hat{p}_i)) < \bar{v}_j \equiv u_j(D_j(\bar{p}_i))$. Then, by stationarity, $\hat{x}_j(\bar{p}_i) \neq$ Reject, i.e., $\hat{x}_j(\bar{p}_i) = D_j(\bar{p}_i)$. Thus, offering $\bar{p}_i$ is a profitable deviation for i, a contradiction. ■

Proof. [Lemma 2]There are two cases.

Case 1: Assume that $\hat{x}_1(\hat{p}_2) \neq$ Reject $\neq \hat{x}_2(\hat{p}_1)$. Then, by Lemma 1.2, for each distinct $i, j \in N$, we have $u_i(w - D_j(\hat{p}_i)) = U_i(u_j(D_j(\hat{p}_i)))$. Since $\hat{x}_j(\hat{p}_i) \neq$ Reject, it follows that the continuation value of i at the beginning of any date $t+1$ at which he makes an offer is $U_i(u_j(D_j(\hat{p}_i)))$. This has two implications. First, since $\hat{p}_j$ is accepted,

$$u_i(D_i(\hat{p}_j)) \geq \delta_i U_i(u_i(D_i(\hat{p}_j))). \tag{15}$$

Second, at t, for any p_j, if $u_i(D_i(p_j)) > \delta_i U_i(u_j(D_j(\hat{p}_i)))$, player i must accept the price p_j and demand $D_i(p_j)$ (see Lemma 1.1). Since j offers $\hat{p}_j$, it must be true that

$$U_j(u_i(D_i(\hat{p}_j))) \geq U_j(v_i) \tag{16}$$

for each $v_i > \delta_i U_i(u_j(D_j(\hat{p}_i)))$. Now, assume that (i) is not true. Then, by (15), $u_i(D_i(\hat{p}_j)) > \delta_i U_i(u_i(D_i(\hat{p}_j)))$. Hence, by (16), $U_j(u_i(D_i(\hat{p}_j))) \geq U_j(v_i)$ for each v_i with $v_i > \delta_i U_i(u_i(D_i(\hat{p}_j)))$. Since $u_i(D_i(\hat{p}_j)) > \delta_i U_i(u_i(D_i(\hat{p}_j)))$, this implies that U_j has a local maximum at $u_i(D_i(\hat{p}_j))$. Thus, by Assumption 2, $u_i(D_i(\hat{p}_j)) = \bar{v}_i$.

Case 2: Assume that $\hat{x}_j(\hat{p}_i) =$ Reject for some $i, j \in N$. If we also had $\hat{x}_i(\hat{p}_j) =$ Reject, agents would never reach an agreement, thus their continuation values would be zero. Since the Walrasian payoffs are strictly positive (by Assumption 1), this would yield a contradiction: In that case, in a stationary equilibrium each agent must accept the Walrasian price p^*, providing a profitable deviation (p^*) for the agent who makes an offer. Therefore, $\hat{x}_i(\hat{p}_j) \neq$ Reject $= \hat{x}_j(\hat{p}_i)$. Then, $\hat{x}_i(p_j) = D_i(p_j)$ if $u_i(D_i(p_j)) > \delta_i^2 u_i(D_i(\hat{p}_j))$. Since j offers $\hat{p}_j$, if $u_i(D_i(\hat{p}_j)) = 0$, then $U_j(v_i) \leq U_j(0)$ for each $v_i > 0$; i.e., $\bar{v}_i = 0 = u_i(D_i(\hat{p}_j))$. On the other hand, if $u_i(D_i(\hat{p}_j)) > 0$, then $u_i(D_i(\hat{p}_j)) > \delta_i^2 u_i(D_i(\hat{p}_j))$. As in Case 1, this yields $u_i(D_i(\hat{p}_j)) = \bar{v}_i$. ■

Proof. [Lemma 3]Assume there exists such $p_i \in P$. Then, if i offers p_i, it will be accepted, yielding a payoff higher than $\bar{v}_i/\delta_i$. Then, at the previous date, i should not accept $\hat{p}_j$, which gives him only $\bar{v}_i$. ■

Proof. [Lemma 4] By Assumption 3, for $\delta_1 = \delta_2 = 1$, $\delta_i U_i(\delta_j U_j(\bar{v}_i)) \geq u_i(w - D_j(p_i)) > \bar{v}_i$. Since U_i is continuous, this implies the Lemma. ■

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